

# Efficient analytical moments for the robustness analysis in design optimisation

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**Abstract:** System uncertainties play a vital role in the robustness (or sensitivity) analysis of system designs. In an iterative procedure such as design optimisation, the robustness analysis that is simultaneously accurate and computationally efficient is essential. Accordingly, the current state-of-the-art techniques such as univariate dimension reduction method (DRM) and performance moment integration (PMI) approach have been developed. They are commonly used to express the sensitivity while utilising the statistical moments of a performance function in an advanced design optimisation paradigm known as the reliability-based robust design optimisation (RBRDO). However, the accuracy and computational efficiency scalability for increasing the problem dimension (i.e. the number of input variables) have not been tested. This study examines the scalability of the above-mentioned pioneering techniques. Additionally, it also introduces a novel analytical method that symbolically calculates the sensitivity of the performance function prior to the iterative optimisation procedure. As a result, it shows a better computational cost scalability when tested on performance functions with increased dimensionality. Most importantly, when applied to real-world RBRDO problems such as the vehicle side impact crashworthiness, the proposed technique is three times faster than the mainstream method while yielding a high quality and safe vehicle design.

## 1 Introduction

In the increasingly complex and competitive modern technological environment, engineers from a wide variety of fields constantly strive toward finding the most economical designs by using the least amount of resources. To find the design parameters that produce the most inexpensive design, a design optimisation procedure is performed whereby an objective function (which is usually the weight or cost) is minimised or maximised with respect to a set of constraint functions. There are various types of evolutionary algorithms readily available in the literature [1, 2] to perform this procedure.

For mission-critical applications where health and safety factors are involved, e.g. the crashworthiness of vehicle design [3], an additional framework of reliability and/or robustness analysis is normally integrated into the optimisation framework. Such analyses are made on the objective and constraint functions by taking into account the system uncertainties. By doing so, the optimisation algorithm searches for the most economical design while the physical constraints are met with a desired level of confidence (or reliability) along with less susceptibility to system uncertainties (due to the inherent robustness analysis). As a result, a higher-quality design that meets the specified safety requirements can be obtained.

The advanced design optimisation paradigm that incorporates both such analyses is known as the reliability-based robust design optimisation (RBRDO) [4, 5]. Though this method is predominantly used in the field of a structural and mechanical design, in the recent years, it has become increasingly popular in the other fields of engineering such as magnetics, manufacturing, microelectronics and micromachining [6–8] etc. due to the increasingly competitive market conditions, stringent safety requirements and the ability of the method to deliver designs that are insensitive to uncontrollable variations [7]. RBRDO is a unified framework that combines two approaches: (a) the reliability-based design optimisation [9] which optimises design objectives for a given set of probabilistic (or reliability) constraints and (b) the RDO [10] which increases the robustness of a designed system by minimising

the sensitivity of the design objectives to process variabilities. Therefore, their integration in RBRDO offers a complete solution that assesses the best compromise between the cost, reliability and robustness. The prime focus of this paper is the robustness (or sensitivity) analysis in the RBRDO procedure.

Though the incorporation of the robustness analysis leads to better quality designs, it incurs high computational load/time. Therefore, such frameworks are often characterised by an inherent trade-off between the accuracy and computational complexity. For instance, Monte Carlo (MC) simulation technique [11] is the state-of-the-art approach in accurately finding the sensitivity of a function. Unfortunately, the use of MC in design optimisation problems imposes a serious computational burden due to the required large sample size as well as due to the iterative nature of the design optimisation process [5, 12]. The first-order Taylor series expansion method [10] is the most simplistic and computationally efficient approach. However, the results can often be inaccurate especially for non-linear functions or input random variables with large variations [5]. Various innovative and efficient numerical techniques such as the moment-based method [4, 5, 12, 13], percentile difference-based method [14] and hybrid quality loss functions-based method [15] have been developed to address the shortcoming. They offer a better trade-off between the accuracy and computational efficiency compared with the MC simulation and Taylor series approximation techniques. On the other hand, the scalability of computational cost and accuracy of the methods have not been tested for problems with a high dimensionality, i.e. with a large number of variables. At the same time, such an analysis is essential as some design problems, e.g. advanced microelectronics package design [6], have a relatively large number of input variables (>5).

This paper investigates the scalability of some of the representative techniques in the moment-based approach for problems with an increased dimensionality. It also introduces a novel analytical moment-based evaluation method that shows a better scalability of the computational cost when tested with the increasing number of random variables (up to 20). The proposed method utilises the polynomial expansion [16] and the mathematical tool [17] that

can precisely calculate the statistical moments of a polynomial function. Its procedural steps (given below in Section 3) allow the moments to be symbolically calculated in advance with the aim of alleviating the computational burden of finding the moments within the iterative optimisation procedure. Section 4 then goes on to show, by comparing against the computationally efficient mainstream moment-estimation techniques, that the proposed analytical method demonstrates a better computational cost scalability with the increase of the problem dimension while maintaining good estimation accuracy. The section also discusses the implication of such a technique for design engineers facing real-world RBRDO problems.

## 2 Overview of the moment-based RBRDO

The traditional optimisation formulation can be represented as (1) whereby  $f_o(\cdot)$  is the objective function to be minimised;  $\mathbf{d}$  is the vector of design variables with lower bounds  $\mathbf{d}^L$  and upper bounds  $\mathbf{d}^U$ ;  $G_l(\cdot)$  is the  $l$ th constraint function; and  $n_c$  is the number of the constraint functions

$$\begin{aligned} & \text{minimise} && f_o(\mathbf{d}), \\ & \text{subject to} && G_l(\mathbf{d}) > 0 \quad l = 1, \dots, n_c \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \end{aligned} \quad (1)$$

In the robust design, the sensitivity analysis is performed on a performance function and the moment-based approach is the most-effective technique in terms of accuracy and computational efficiency [5, 18]. A performance function (represented as  $h(\mathbf{X})$ , where  $\mathbf{X}$  denotes the vector of random variables) is a mathematical model that quantifies the performance of a system-under-design. Theoretically, the statistical moments of  $h(\mathbf{X})$ ,  $E[h^k(\mathbf{X})]$  can be calculated using the multi-dimensional integral (2), where  $k$  denotes the order of the statistical moment;  $\mathbf{x}$  denotes realisations of the random variables  $\mathbf{X}$ ; and  $f_X(\mathbf{x})$  is the joint probability density function of  $\mathbf{X}$

$$E[h^k(\mathbf{X})] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h^k(\mathbf{X}) f_X(\mathbf{x}) d\mathbf{x} \quad (2)$$

In the moment-based RBRDO, the objective is to simultaneously minimise the discrepancy of the mean  $\mu_h = E[h(\mathbf{X})]$  and variance

$\sigma_h^2 = E[h^2(\mathbf{X})] - (E[h(\mathbf{X})])^2$  of the performance function  $h(\cdot)$  with respect to the design specifications. The objective function  $f_o(\cdot)$  is therefore modified to be expressed in terms of  $\mu_h$  and  $\sigma_h^2$ :  $f_o(\mu_h, \sigma_h^2)$ . The reliability of the design, however, is ensured with a probabilistic constraint  $\Pr[G_l(\mathbf{X}, \mathbf{d}) > 0] \leq \Phi(-\beta_l)$  whereby  $\Phi(\cdot)$  is the normal inverse cumulative distribution function;  $\Pr[\cdot]$  is the probability operator;  $\mathbf{X}$  is the vector of random variables; and  $\beta_l$  is the  $l$ th targeted reliability level. If  $\Pr[G_l(\mathbf{X}, \mathbf{d}) > 0]$  is more than the constraint  $\Phi(-\beta_l)$ , then the design is deemed non-compliant and a different set of design points are searched by the optimisation algorithm.

By incorporating these conditions with formulation (1), a typical moment-based RBRDO problem can be mathematically represented as (3). Here, the vector  $\mathbf{d}$  maybe either independent or be linked to the random variable vector  $\mathbf{X}$ . In most design problems  $\mathbf{d} = E[\mathbf{X}]$ , where  $E[\mathbf{X}]$  is the expectation of the random variable  $\mathbf{X}$

$$\begin{aligned} & \text{minimise} && f_o(\mu_h, \sigma_h^2), \\ & \text{subject to} && \Pr[G_l(\mathbf{X}, \mathbf{d}) > 0] \leq \Phi(-\beta_l) \quad l = 1, \dots, n_p \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \end{aligned} \quad (3)$$

This paper introduces an analytical approach for calculating the mean  $\mu_h$  and variance  $\sigma_h^2$  of a very general class of continuous performance functions in Section 3. The method presented in this paper is the analytical moment-based RBRDO.

There are various advanced techniques presented in the literature to perform the inherent sensitivity analysis in RBRDO procedure. Apart from the well-known MC simulation and first-order Taylor series approximation techniques, they can be categorised into three main groups: (i) moment-based [4, 5, 12, 13]; (ii) percentile difference-based [14]; and (iii) hybrid quality loss functions-based [15]. The research [5, 18] show that the moment-based approach is simple, reliable and computationally efficient compared with the other approaches. For example, the research [5] shows that the percentile difference-based method provides inconsistent variance estimation (of the performance function) depending on the percentile locations used, whereas the paper [18] shows that the hybrid quality loss functions-based method is computationally more expensive than the moment-based one. For this reason, the effectiveness of the proposed analytical moment-based technique will be benchmarked (in Section 4 below) against the two most prominent mainstream moment-based approaches: the univariate DRM [5, 13] and the PMI [12] method.

In the univariate DRM method, the performance function  $h(\cdot)$  is additively decomposed into one-dimensional (1D) functions and the moment-based integration rule [13] is then applied to numerically calculate the multi-dimensional integral (2). The PMI method, on the other hand, reduces (2) into 1D integral using Rosenblatt transformation [19] and numerically calculates it using the first-order reliability method [20]. In both the techniques, the dimensionality of the problem is first reduced in effort to boost the computational efficiency. The moments,  $E[h(\cdot)]$  and  $E[h^2(\cdot)]$ , are then numerically calculated utilising the dimensionally reduced problem and  $n_q$  number of quadrature points. The significance of  $n_q$  as well as the underlying theoretical framework of these methods are discussed in a greater depth in [5, 12, 13].

## 3 Analytical moment-based RBRDO

Section 2 presented the general formulation of the RBRDO methodology. It also briefly described the commonly used univariate DRM and PMI methods to find the mean and variance of the performance function  $h(\cdot)$ . This section presents the new analytical moment-based RBRDO (Fig. 1). Here, by utilising the exact analytical moment expressions (obtained by using the moment calculator [17]), the computational burden of finding the statistical moments of  $h(\cdot)$  within the optimisation loop is minimised. As a result, the

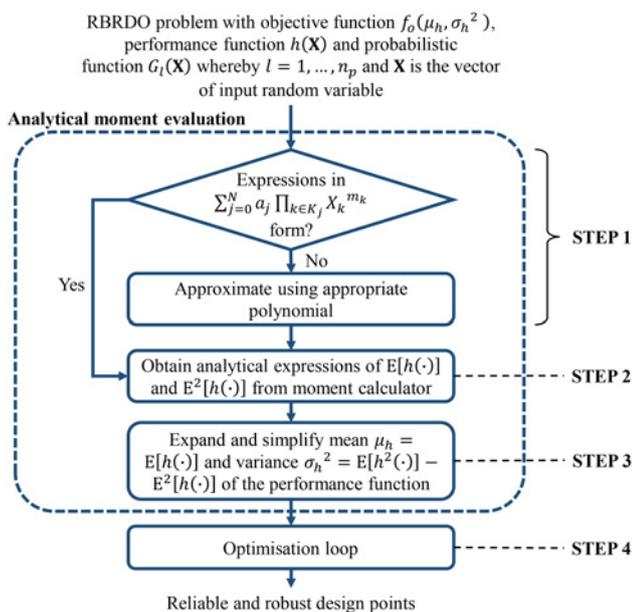


Fig. 1 Proposed analytical moment-based RBRDO procedure

overall computational burden of the analytical moment-based RBRDO is significantly diminished without compromising the accuracy of the calculated moments.

The new analytical moment-based RBRDO procedure illustrated in Fig. 1 is implemented in four systematic steps:

*Step 1:* Represent  $h(\cdot)$  and  $h^2(\cdot)$  in the form  $\sum_{j=0}^N a_j \prod_{k \in K_j} X_k^{m_k}$  in order to employ the moment calculator. This form is a common end-result of multivariate polynomial approximation models or response surface methodology [21] from empirical data and a numerical solution to dynamical equations. If the function  $h(\cdot)$  is not given in this required form, it can be analytically approximated by employing the polynomial dimensional decomposition method [22] or power series expansion such as the Taylor and Laurent series [16, 23]. This paper uses the Taylor series expansion and a brief discussion on this algorithm step is presented below in Section 3.2.

*Step 2:* Employ the moment calculator to precisely calculate  $E[h(\cdot)]$  and  $E[h^2(\cdot)]$ . Section 3.1 below demonstrates the use of the calculator.

*Step 3:* Simplify the calculated expressions for  $E[h(\cdot)]$  and  $E[h^2(\cdot)]$  using the expressions  $\mu_h = E[h(\cdot)]$  and  $\sigma_h^2 = E[h^2(\cdot)] - (E[h(\cdot)])^2$  for the mean and variance of the performance function, respectively. Since the evaluation of the moments is done analytically, steps 1–3 are only needed to be performed once before the optimisation loop begins.

*Step 4:* Determine the type of objective function  $f_o(\mu_h, \sigma_h^2)$ . Then, formulate and solve the RBRDO problem using any optimisation algorithm and reliability analysis method. This paper uses the well-known performance measure approach (PMA) [24, 25], which is discussed below in Section 3.3.

The proposed analytical moment-based RBRDO method assumes the independence of random variables. This is the same assumption as that made in the other mainstream moment-estimation techniques discussed in this paper. In cases where correlated variables are involved, the transformation can be used to map these dependent variables into an equivalent set of independent variables [26].

### 3.1 Analytical moments using moment calculator

The use of the Mellin transform to obtain precise moments of any polynomial was proposed and validated in [27, 28]. An open-access-type moment calculator was also developed [28] to enable an automatic simplification of the algebraic operations. To provide a quick demonstration, the procedure of obtaining the mean and variance using the calculator is presented below by employing a simple example.

Let the performance function  $h(\mathbf{X})$  have the following general type (4)

$$h(\mathbf{X}) = \sum_{j=0}^N a_j \prod_{k \in K_j} X_k^{m_k}. \quad (4)$$

Moreover, let the particular form of the performance function be  $h(\mathbf{X}) = X_1^2 + X_1 X_2$ . The expressions for  $E[h(\mathbf{X})]$  and  $E[h^2(\mathbf{X})]$  are sought. Following the step 1 of the procedure,  $h^2(\mathbf{X}) = X_1^4 + X_1^2 X_2^2 + 2X_1^3 X_2$ . Since both  $h(\mathbf{X})$  and  $h^2(\mathbf{X})$  are in the required form (4), there is no any need here for the approximation. Let both the random variables follow the normal distribution with a mean  $\mu_X$  and standard deviation  $\sigma_X$ . The application of the moment calculator yields  $E[h(\mathbf{X})] = \mu_{X_1}^2 + \mu_{X_1} \mu_{X_2} + \sigma_{X_1}^2$  and  $E[h^2(\mathbf{X})] = \mu_{X_1}^4 + 6\mu_{X_1}^2 \sigma_{X_1}^2 + 3\sigma_{X_1}^4 + 2\mu_{X_2} (\mu_{X_1}^3 + 3\mu_{X_1} \sigma_{X_1}^2) + (\mu_{X_1}^2 + \sigma_{X_1}^2) (\mu_{X_2}^2 + \sigma_{X_2}^2)$ . Finally, both expressions are substituted following the step 4 to obtain the

mean and variance of performance function  $h(\mathbf{X})$  as  $\mu_{h(\mathbf{X})} = \mu_{X_1}^2 + \mu_{X_1} \mu_{X_2} + \sigma_{X_1}^2$  and  $\sigma_{h(\mathbf{X})}^2 = \sigma_{X_1}^2 (2\mu_{X_1} + \mu_{X_2})^2 + 2\sigma_{X_1}^4 + \sigma_{X_2}^2 (\mu_{X_1}^2 + \sigma_{X_1}^2)$ , respectively.

By using the moment calculator, the final expressions for  $E[h(\mathbf{X})]$  and  $E[h^2(\mathbf{X})]$  are obtainable in a single step given that  $h(\mathbf{X})$  is in the form (4). Both expressions have been proven to be analytically valid regardless of values of the distribution parameters [27].

In the above example, the parameters are the mean  $\mu_X$  and standard deviation  $\sigma_X$ . Typically, the complexity (or length) of the output expression increases as the non-linearity of the input performance function (or the number of random variables) is getting higher. However, this complexity is of a minor concern, as it has minimal impact on the execution speed by a modern commodity computer (this is shown in Section 4.2).

As mentioned above, the online tool is directly applicable when the performance function naturally takes the form (4). In practice, however, not all performance functions are represented in such a form. To address it, the use of the power series expansion (outlined in Section 3.2) is incorporated into the RBRDO framework (Fig. 1).

### 3.2 Power series expansion

The online moment calculator enables finding  $E[h^k(\cdot)]$  regardless of the value of  $k$ . The calculator requires that the relationship between  $h(\cdot)$  and the input variables  $\mathbf{X}$  take the polynomial form (4). For performance functions that do not satisfy this condition, the terms in expression  $h^k(\cdot)$  should be approximated by an appropriate polynomial. Owing to the space constraint, only univariate Taylor's series approximation is discussed in this paper. More sophisticated multivariate polynomial approximation (such as the multivariate Taylor series) can be employed for multivariate performance functions.

For example, let  $g(\cdot)$  be an analytical function of an arbitrary order  $m$ . The higher-order function  $g^m(X_i)$  can be derived by performing the following steps:

- For  $X_i \sim \mu_i + \xi_i$ , where  $\mu_i$  is an appropriate translation parameter (usually the mean of variable  $X_i$ ), the Taylor series of  $g^m(X_i)$  around  $\mu_i$  is written as  $\sum_{n=0}^{\infty} (\{g^m\}^{(n)}(\mu_i)/n!) (X_i - \mu_i)^n$ .
- After simplification, the series is  $g^m(X_i) = \sum_{n=0}^{\infty} c_n \xi_i^n$ , whereby  $c_n$  are known constants and  $\xi_i^n = X_i - \mu_i$ . This form is consistent with (4). To employ the calculator, the series is expanded to a finite (but sufficient) number of terms.
- Finally, the new function is created to analytically approximate  $h^k(\cdot)$ , and it is compatible with the moment calculator.

For illustration, let a given performance function is  $h(\mathbf{X}) = X_1^2 + \exp(X_2)$ . Here, the second right term [ $\exp(X_2)$ ] is not in the required form. Hence, it is represented using the Taylor series as  $h(\mathbf{x}) = X_1^2 + \sum_{n=0}^{n_T} (1/n!) X_2^n$  for  $n_T$  number of terms. This allows the moment calculator to be deployed to find the moments of  $h(\mathbf{X})$  (as shown in Section 3.1).

### 3.3 Reliability analysis and optimisation loop

The final step of the analytical moment-based RBRDO procedure is a design optimisation through an iterative search. Since the proposed analytical method is only concerned with the evaluation of the objective function (specifically, the moment evaluation of the performance function  $h(\cdot)$ ), it is compatible with any off-the-shelf optimisation routines and reliability analysis methods. Therefore, the choice of the optimisation algorithm and reliability analysis method is immaterial for the purpose of this paper.

The algorithm chosen for the reliability analysis in this paper is the well documented and validated PMA [24, 25]. In PMA, the reliability analysis of an arbitrary  $l$ th probabilistic function  $G_l(\cdot)$  in (3) is performed using the technique known as the inverse reliability analysis

[25]. There most probable point (MPP) is the failure point that corresponds to the lowest performance level that satisfies the target reliability index  $\beta_t$ . A specific algorithm known as the conjugate gradient analysis method [29] is employed in this paper to perform the reliability analysis.

#### 4 Examples and discussion

This section experimentally compares and analyses the effectiveness of applying the proposed analytical method based on the following three criteria:

- variance estimation accuracy and efficiency;
- scalability when increasing a number of random variables in performance functions; and
- implication of the proposed method on real-world RBRDO designs.

To keep the discussion succinct, only the variance of performance functions is calculated as it comprises both the first- and second-order statistical moments. The analytical method is benchmarked against the two most effective and computationally efficient moment-based methods: univariate DRM [5] and PMI [12].

##### 4.1 Variance estimation accuracy and efficiency

In the following two numerical problems, different types of functions (concave and convex) are used to test the accuracy and efficiency of the analytical moment-based method in calculating the variance. Research [25] shows that some methods maybe ineffective for either concave or convex function. Thus, it is particularly important to ensure that the analytical method is tested using these two types of functions to validate its effectiveness.

**4.1.1 Problem setup:** In this example, the performance functions  $h_1(\mathbf{X})$  and  $h_2(\mathbf{X})$  shown in (5) and (6) are concave and convex functions, respectively, whereas  $I_0(\cdot)$  in (6) denotes the Bessel function of the first kind [30]. Input random variables  $X_1$ ,  $X_2$  and  $X_4$  are normally distributed with the means {4.0, 5.0, 6.0}, respectively, and standard deviation of 0.8. The random variable  $X_3$  is assumed to follow the uniform distribution within the limit [4.8, 7.2]. The goal here is to calculate the variance of  $h_1(\mathbf{X})$  and  $h_2(\mathbf{X})$

$$h_1(\mathbf{X}) = \frac{\exp(0.8X_1 - 1.2) + \exp(0.7X_2 - 0.6) - 5}{10} \quad (5)$$

$$h_2(\mathbf{X}) = -I_0(X_3 - 7) - X_4 + 10 \quad (6)$$

Following the proposed analytical moment-based method (Fig. 1), variance values of the performance functions are evaluated. The number of terms used for the Taylor series expansion is  $n_T = 3$ ,  $n_T = 5$  and  $n_T = 7$ . It is important to note that this step is performed only once in RBRDO because the same analytical expressions can be used multiple times with different values of  $\mu_X$  and  $\sigma_X$  in the design optimisation. On the other hand, the DRM and PMI methods are implemented for different numbers of the quadrature points:  $n_q = 3$ ,  $n_q = 5$  and  $n_q = 7$ . The results are compiled in Table 1.

**4.1.2 Performance metrics and results:** The percentage error of the variance estimation as well as the equivalent function evaluation (EFE) are used as the performance metrics in this paper. The percentage error is calculated based on the difference between the estimated variance and the variance obtained from the numerical integration (NI) of the integral (2). The expression  $(n_q - 1) \times n_r + 1$  and the number of performance function calls are used to determine the EFE for the univariate DRM and PMI methods, respectively [5]. EFE reflects the computational efficiency of an approach whereby a smaller EFE corresponds to a better computational efficiency.

Since the moment computation is done before the optimisation loop, to form a fairer assessment, the EFE for the analytical approach is reported based on the median computational time for the expressions of  $\sigma_{h_1(\mathbf{X})}^2$  and  $\sigma_{h_2(\mathbf{X})}^2$  with reference to  $h_1(\mathbf{X})$  and  $h_2(\mathbf{X})$ , respectively. For example, the median execution time of  $10^8 h_1(\mathbf{X})$  is 20.47 s while  $10^8 \sigma_{h_1(\mathbf{X})}^2$  for  $n_T = 3$  takes 44.01 s. Thus, the complexity factor of  $\sigma_{h_1(\mathbf{X})}^2$  is 2.15, and every evaluation of  $\sigma_{h_1(\mathbf{X})}^2$  is counted as equivalent to 2.15  $h_1(\mathbf{X})$  function evaluation. In other words, EFE to evaluate one  $\sigma_{h_1(\mathbf{X})}^2$  using the analytical method is 2.15.

**4.1.3 Observations and discussion:** Table 1 shows that both the PMI and DRM approaches provide a fair estimation of the variances of  $h_1(\mathbf{X})$  and  $h_2(\mathbf{X})$  when compared against the NI. Moreover as anticipated, the EFE counts for the DRM and PMI methods increase with the number of quadrature points  $n_q$  thus showing that the higher computational cost is associated with more quadrature points. However, the effect to the variance estimation accuracy is different for each method. For instance, increasing the number of quadrature points did not guarantee an increase in the variance estimation accuracy for the PMI method as it does for the DRM.

The proposed analytical method also provides a fair estimation of  $\sigma_{h_1(\mathbf{X})}^2$  and  $\sigma_{h_2(\mathbf{X})}^2$ . Increasing the number of Taylor series expansion terms gives an increasingly more accurate moment estimation. Most notably, the analytical method gives a very good trade-off between the accuracy and computational load. For example, the analytical method with  $n_T = 7$  is capable of estimating the variances with almost 0% error. Additionally, for the similar level of accuracy, it consistently maintains a comparable computational load with the most-efficient mainstream moment-based method: the univariate DRM. However, note that the performance functions (5) and (6) are low in dimensionality, and it is important for the analytical method to exhibit a similar trade-off even for high-dimensional problems. Therefore, Section 4.2 examines the scalability of computational efficiency as well as the accuracy for performance functions with a higher number of variables (up to 20).

##### 4.2 Scalability when increasing the number of random variables in performance functions

Normally, the computational cost to calculate the moments increases exponentially with a dimension of a problem (the number of random variables). This is known as the curse of dimensionality [22]. To investigate the computational efficiency scalability of the analytical method, this section uses performance functions with the increasing dimensionality up to 20 random variables.

**4.2.1 Problem setup:** The previous two examples have analysed the performance (i.e. accuracy and efficiency) of the analytical method with simple performance functions with a low dimensionality. Here, the functions  $h_3(\mathbf{X})$  and  $h_4(\mathbf{X})$  given by (7) and (8), respectively, are used to test the computational efficiency scalability with the increasing function dimensionality. Expression (7) is an exponential function with random variables following the uniform distribution within the limit [-1, 1]. Expression (8) is a polynomial function with random variables following the normal distribution with the mean equal to 1 and the standard deviation of 0.4. The variances of  $h_3(\mathbf{X})$  and  $h_4(\mathbf{X})$  are calculated for the set of dimensions  $s \in \{4, 8, 12, 16, 20\}$ . The number of terms used in the Taylor series expansion for  $h_3(\mathbf{X})$  is  $n_T = 10$ . The conventional methods are implemented with  $n_q = 3$

$$h_3(\mathbf{X}) = \exp\left(\prod_{i=1}^s X_i\right) \quad s \in \{4, 8, 12, 16, 20\} \quad (7)$$

$$h_4(\mathbf{X}) = 1 + \sum_{i=1}^s \prod_{j=1}^i X_j \quad s \in \{4, 8, 12, 16, 20\} \quad (8)$$

**4.2.2 Performance metrics and results:** Similar to the previous example, the EFE and percentage errors of the calculated variances

**Table 1** Comparison of a variance of the concave performance function (5) and convex performance function (6) using the PMI, DRM, analytical and the NI methods

Performance metrics	Moment-estimation techniques									
	PMI			DRM			Analytical			NI
	$n_q = 3$	$n_q = 5$	$n_q = 7$	$n_q = 3$	$n_q = 5$	$n_q = 7$	$n_T = 3$	$n_T = 5$	$n_T = 7$	
variance, $\sigma_{h_1(X)}^2$	1.7820	1.8553	1.8565	1.8786	1.9767	1.9774	1.4677	2.0198	2.0772	2.0811
error, %	14.02	10.85	10.79	9.74	5.02	4.98	29.47	2.95	0.18	–
EFE	22.00	50.00	88.00	10.00	18.00	26.00	2.15	5.34	8.49	–
variance, $\sigma_{h_2(X)}^2$	0.9150	0.9209	0.9217	0.8554	0.8573	0.8573	0.7650	0.8463	0.8566	0.8573
error, %	6.73	7.21	7.33	0.22	0.00	0.00	10.77	1.28	0.08	–
EFE	40.00	78.00	108.00	10.00	18.00	26.00	2.06	6.51	17.10	–

**Table 2** Comparison of a variance of the performance functions (7) and (8) for  $s = 4$  and 8 using the analytical method and the MC simulation results with  $10^7$  samples

	Variance, $\sigma_{h_3(X)}^2$		Variance, $\sigma_{h_4(X)}^2$	
	4	8	4	8
number of random variables, $s$				
MC simulation	5.3409	42.955	0.0133	0.0002
analytical	5.3413	42.999	0.0133	0.0002

are used as the performance metrics. The EFE calculation is done in the same way as discussed above in Section 4.1 for all the methods. As the dimension of the performance function increases ( $s > 8$ ), the NI of (2) becomes impractical while the convergence of the MC simulation requires unaffordable computational time: the performance function with 20 independent variables would require at least  $(10^4)^{20}$  samples to be reliable. Thus, the analytical method is practically the only viable approach to calculate the variance of high-dimensional functions. Table 2 shows that the variance estimated using the analytical method agrees with the MC simulation results for  $s = 4$  and 8. Besides, it was proven to be mathematically correct in the earlier works [27, 28], and was demonstrated numerically in Section 4.1. Thus, Fig. 2a presents the EFE with increasing dimensionality for all the methods, while Fig. 2b reports the error of the PMI and DRM approaches relative to the analytical method.

**4.2.3 Observation and discussion:** The results shown in Fig. 2a are well in agreement with the literature [5] stating that the PMI method becomes computationally more efficient for high-dimensional problems. The EFE of DRM scales linearly with respect to the number of variables, whereas the EFE for the PMI method appears to be insensitive on the problem dimension. The analytical method incurs the lowest computational cost and is characterised by the slowest growth of the computational complexity (or EFE) with respect to the problem dimension among all the three compared methods. In short, the analytical method has the best scalability with respect to the increasing problem dimension. The

implication of such an advantage on a real-world RBRDO design scenario is discussed in Section 4.3 below.

It can be observed in Fig. 2b that both the univariate DRM and PMI reach 100% error for the variance of  $h_3(X)$  at  $s \geq 5$ . Both methods wrongly estimate the variance of  $h_3(X)$  as 0 in this case – as shown in Table 2, the variance of  $h_3(X)$  gets smaller with the increasing number of variables  $s$ . For  $h_4(X)$  however, the variance estimation error deteriorates with the increase of the dimension. Nevertheless, for functions  $h_3(X)$  and  $h_4(X)$ , both the methods do not perform well with the error ranging anywhere between 10 and 100% for  $s > 5$ . The root cause for such an effect in the PMI method is the MPP search which becomes increasingly unreliable when a large number of variables are present. In the DRM method, it is caused by the errors introduced in the univariate decomposition for highly non-linear functions. The impact of such shortcomings on a real-world RBRDO problem is demonstrated in the subsequent section.

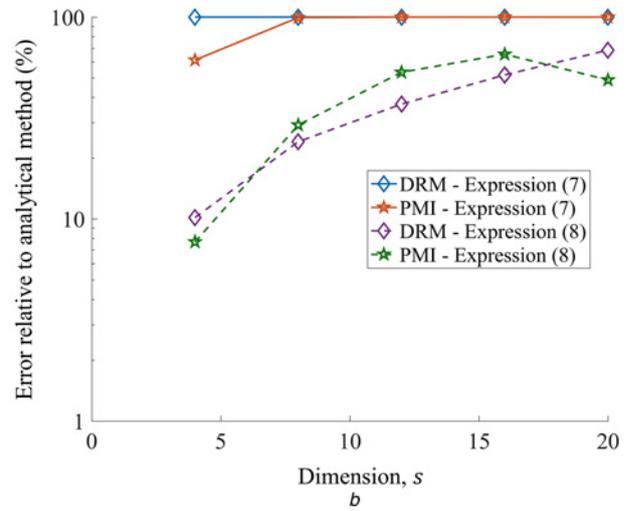
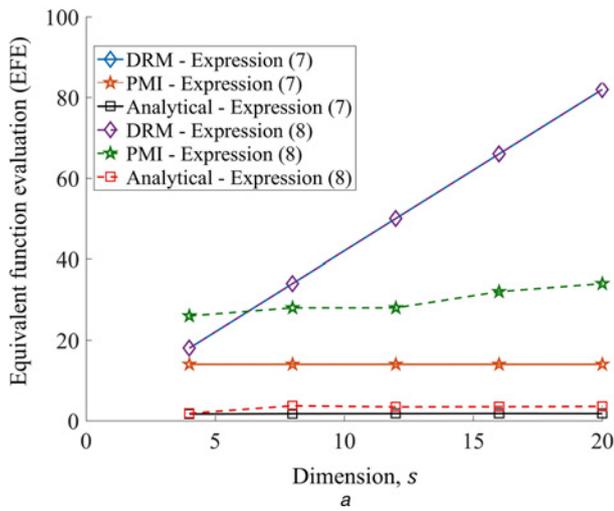
### 4.3 Implication of the proposed analytical method on real-world RBRDO designs

In this section, a real-world optimisation problem is considered – it is a vehicle side impact crashworthiness [3]. The design objective is to minimise the weight of the vehicle  $W(X)$  as well as the variance of the performance function  $H(X)$  while enhancing the side impact crash protection to ensure the safety of passengers. To do so, the European Enhanced Vehicle-Safety Committee side impact procedure [31] is used in order to determine the reliability constraints of the performance functions.

**4.3.1 Problem setup:** In this paper, the vehicle side impact crashworthiness (which is well documented in [3]) is formulated as (see (9))

whereby  $f_o(X)$  is the objective function.  $W_0(X) = 29.05$  kN is the initial weight and  $H_0(X) = 1.4781$  mm is the initial variance of the performance function based on the original design points given in [3].

$$\begin{aligned}
 &\text{minimise} && f_o(\mathbf{X}) = w_1 \frac{W(\mathbf{X})}{W_0(\mathbf{X})} + w_2 \frac{\sigma_{H(\mathbf{X})}^2}{\sigma_{H_0(\mathbf{X})}^2}, \\
 &\text{subject to} && \Pr[\text{abdomen load} > 1 \text{ kN}] \leq \Phi(-\beta) \\
 &&& \Pr[\text{upper/middle/lower viscous criteria} > 0.32 \text{ m/s}] \leq \Phi(-\beta) \\
 &&& \Pr[\text{upper/middle/lower rib deflection} > 32 \text{ mm}] \leq \Phi(-\beta) \\
 &&& \Pr[\text{pubic symphysis force} > 4 \text{ kN}] \leq \Phi(-\beta) \\
 &&& \Pr[\text{velocity of B – pillar at middle point} > 9.9 \text{ mm/ms}] \leq \Phi(-\beta) \\
 &&& \Pr[\text{velocity of front door at B – pillar} > 15.7 \text{ mm/ms}] \leq \Phi(-\beta) \\
 &&& d^L \leq d \leq d^U \text{ and } \beta = 2,
 \end{aligned} \tag{9}$$



**Fig. 2** Results of  
a EFE

b Error relative to the analytical method in estimating the variance of performance functions  $h_3(X)$  and  $h_4(X)$  in (7) and (8), respectively. Both (7) and (8) are analytic functions with increasing dimensionality. The variance estimation error is calculated relative to the analytical method

Table 3 provides the expressions for the probabilistic functions. The initial design points, statistical information, as well as details on the random variables are given in [3]. Note that  $X_{10}$  and  $X_{11}$  are not regarded as design variables in this problem, and the weight  $W(X)$  in Table 3 is given in terms of  $d = E[X]$ . The target reliability index for all probabilistic functions is  $\beta = 2$ . The lower rib deflection is the performance function  $H(X)$  in this RBRDO problem to represent safety component of the design objective. Since  $H(X)$  is already in the required form, the online calculator [17] can be employed without the need for any analytical transformation in order to calculate the first-order moment  $E[H(X)]$  and second-order moment  $E[H^2(X)]$ . The univariate DRM and PMI method are implemented with  $n_q = 3$ .

**4.3.2 Performance metrics and results:** In addition to the EFE, the actual objective function  $f_o(X)$  is also used as one of the performance metrics. The method to calculate EFE remains the same as in Sections 4.1 and 4.2, while the actual variance of the performance function and the actual objective function for each design is calculated using MC simulation with  $10^7$  samples. The RBRDO

method here is implemented using MATLAB's *fmincon* function [32] and solved using its inbuilt sequential quadratic programming algorithm with the tolerance of the function's termination condition of  $10^{-6}$ , the tolerance on constraint violation of  $10^{-6}$  and the maximum iteration number of 400. The final design outputs are compiled and tabulated in Table 4.

**4.3.3 Observation and discussion:** Table 4 shows that each RBRDO technique yields different optimal design points with respect to the side impact crashworthiness. All the employed three methods show the significant reduction in the objective function  $f_o(X)$  (which is a function of the vehicle weight  $W(X)$ ) as well as the variance of the lower rib cage deflection of the occupants. In addition, all the methods yield design points with probabilistic functions that meet the minimum probabilistic constraints of  $\beta = 2$ . Hence, regardless of the chosen design, the vehicle will meet the high safety standards. However, the proposed analytical method can achieve it with at least three times faster computational speed than the mainstream methods of estimating the variance  $\sigma_{H(X)}^2$  in the optimisation process.

**Table 3** Expressions of objective and performance functions whereby  $d = E[X]$

Description	Expression
weight, $W(X)$	$1.98 + 4.9d_1 + 6.67d_2 + 6.98d_3 + 4.01d_4 + 1.78d_5 + 2.73d_7$
abdomen load	$1.16 - 0.3717X_2X_4 - 0.00931X_2X_{10} - 0.484X_3X_9 + 0.01343X_6X_{10}$
lower rib deflection, $H(X)$	$46.36 - 9.9X_2 - 12.9X_1X_8 + 0.1107X_3X_{10}$
middle rib deflection	$33.86 + 2.95X_3 + 0.1792X_{10} - 5.057X_1X_2 - 11X_2X_8 - 0.0215X_5X_{10} - 9.98X_7X_8 + 22X_8X_9$
upper rib deflection	$29.98 + 3.818X_3 - 4.2X_1X_2 + 0.0207X_5X_{10} + 6.63X_6X_9 - 7.7X_7X_8 + 0.32X_9X_{10}$
upper viscous criteria	$0.261 - 0.0159X_1X_2 - 0.188X_1X_8 - 0.019X_2X_7 + 0.0144X_3X_5 + 0.0008757X_5X_{10} +$ $0.08045X_6X_9 + 0.00139X_8X_{11} + 0.00001575X_{10}X_{11}$
middle viscous criteria	$0.214 + 0.00817X_5 - 0.101X_1X_8 - 0.0704X_1X_9 + 0.03099X_2X_6 - 0.018X_2X_7 + 0.0208X_3X_8 + 0.121X_3X_9 - 0.00364X_5X_6 +$ $0.0007715X_5X_{10} - 0.0005354X_6X_{10} + 0.00121X_8X_{11} + 0.00184X_9X_{10} - 0.02X_2^2$
lower viscous criteria	$0.74 - 0.61X_2 - 0.163X_3X_8 + 0.001232X_3X_{10} - 0.166X_7X_9 + 0.277X_2^2$
pubic symphysis force	$4.72 - 0.5X_4 - 0.19X_2X_3 - 0.0122X_4X_{10} + 0.009325X_6X_{10} + 0.000191X_{11}^2$
velocity of B-pillar at middle point	$10.58 - 0.674X_1X_2 - 1.95X_2X_8 + 0.02054X_3X_{10} - 0.0198X_4X_{10} + 0.028X_6X_{10}$
velocity of front door at B-pillar	$16.45 - 1.489X_3X_7 - 0.843X_5X_6 + 0.0432X_9X_{10} - 0.0556X_9X_{11} - 0.000786X_{11}^2$

**Table 4** Performance of the PMI, DRM and analytical method in the vehicle side impact crashworthiness evaluation

Employed method	Optimum design				EFE
	Actual objective function, $f_o(X)$ from MC	Weight, $W(X)$ , kN	Variance of $H(X)$ , $\text{mm}^2$	Actual variance of $H(X)$ from MC, $\text{mm}^2$	
PMI, $n_q = 3$	0.7069	28.7780	0.0618	0.6254	4590.00
DRM, $n_q = 3$	0.6024	25.8685	0.4144	0.4645	4940.00
analytical	0.5987	25.9044	0.4511	0.4518	1526.07

Furthermore, Table 4 shows that the proposed analytical method has achieved the lowest optimum function  $f_o(X)$  in comparison with the mainstream moment-based methods. The accuracy of the variance estimation using the proposed method indicates that the balance between the vehicle weight and the safety has been achieved as envisioned by the designer. On the other hand, an under-estimated variance (by the univariate DRM for this example) will lead to a low-weight design at the expense of vehicle safety. Even the designer might be unaware of the elevated risk because the reported variance is  $0.4144 \text{ mm}^2$  while the true value is  $0.4645 \text{ mm}^2$ . Though a more accurate estimation maybe achieved by either increasing the quadrature points  $n_q$  used or by employing the bivariate DRM, doing so will, in turn, increase the EFE count.

It is equally important for a moment evaluation technique to be accurate and computationally efficient in a design optimisation procedure. Hence, the use of the proposed analytical method is a preferable option as it meets both the criteria for this RBRDO problem. Moreover, the analytic approach involves a straightforward application of the moment calculator to get an expression for moments without needing to worry about the degree of the function non-linearity, and the problem dimensionality. This could be a very important advantage from practicing design engineers' perspectives.

## 5 Conclusion

The RBRDO is a unified methodology that yields the superior design solutions while offering the best compromise between the cost, reliability and robustness of a system-under-design. The difficulty in overcoming the mathematical complexity in analytically evaluating the statistical moments of the performance function  $h(\cdot)$  in the moment-based RBRDO has led to the development and implementation of various numerical moment-estimation techniques such as the univariate DRM and PMI technique. However, both of them have come with the accuracy versus computational efficiency trade-off, significance of which has not been tested for problems with large number of variables ( $>5$ ).

This paper examines the computational cost and accuracy scalability of the representative mainstream moment-based techniques for higher problem dimensions. Also, it introduces a new analytical moment-based RBRDO framework to derive the high-order moments of a specific functional form using the Mellin transform in conjunction with the analytical approximation techniques employing power series expansion.

For problems with a small number of variables (e.g. 2), the performed experimental study shows that the proposed analytical approach provides equally accurate results while being computationally as efficient as the competing state-of-the-art methods.

However, when tested on problems with a higher number of random variables (up to 20), the proposed analytical approach provides a better scalability with respect to the dimension of the performance function – e.g. it remains computationally efficient. This was demonstrated on the vehicle side impact crashworthiness problem whereby the proposed method was capable of finding the best optimal design while being at least three times faster than the mainstream DRM and PMI methods without sacrificing the design quality. The new analytical method decouples the computationally expensive moment-estimation process from the iterative optimisation loop, thus facilitating higher-quality design solutions at a lower computational cost.

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